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SOME FIXED POINT RESULTS IN b-METRIC SPACES

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Abstract: In this paper, we show that different type of contraction mappings have unique fixed point in b-metric spaces.

Introduction and Preliminaries: The concept of b-metric space was introduced by Bakhtin in [1] and used by Czerwik in [6].

It is well known that Banach's contraction mapping theorem is one of the pivotal results of functional Analysis. A mapping $T: X \to X$ where (X, d) is a metric space, is said to be a contraction if there exists $k \in [0,1)$ such that $\forall\, x, y \in X$

$$d(Tx, Ty) \leq kd(x, y) \qquad \text{----------(1.1)}$$

If the metric space (X, d) is complete the mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of T. We have some contractive condition which will imply existence of fixed point in a complete metric space but will not imply continuity (See [10]).

In this paper, we establish some new contractive type condition for mappings defined on b-metric spaces and prove some new fixed point theorems for these mappings. Our results are generalizations of results in [10].

Definition 1[1]: Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to R_+$, is called a b-metric provided that, for all $x, y, z \in X$,

1) $d(x, y) = 0$ iff $x = y$,
2) $d(x, y) = d(y, x)$,
3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space. It is clear that definition of b-metric space is a extension of usual metric space.

Example 1[7]: The space $l_p (0 < p < 1)$,

$$l_p = \left\{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\right\},$$

together with the function $d: l_p \times l_p \to \mathbb{R}$

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$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}},$$

where $x = x_n, y = y_n \in l_p$ is a b-metric space. By an elementary calculation we obtain that

$$d(x,z) \leq \frac{1}{2^p}[d(x,y) + d(y,z)]$$

Example 2[7]: The $L_p(0 < p < 1)$ of all real functions $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$,
is b-metric space if we take

$$d(x,y) = \left[\int_0^1 |x(t) - y(t)|^p dt\right]^{\frac{1}{p}},$$

for each $x, y \in L_p$.

Definition 2[7]: Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition 3[7]: Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called convergent sequence if and only if there exists $x \in X$ such that for all there exists $n(\varepsilon) \in N$ such that for all $n \geq n(\varepsilon)$we have $d(x_n, x) < \varepsilon$. In this case, we write $lim_{n\to\infty} x_n = x$.

Definition 4[7]: The b-metric space is complete if every Cauchy sequence convergent.

Definition 5[9]: Let E be a non-empty set and $T: E \to E$ a self map. We say that $x \in E$ is a fixed point of T if $T(x) = x$ and denote by FT or $Fix(T)$ the set of all fixed points of T.

Let E be any set and $T: E \to E$ a self map. For any given $x \in E$ we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$, we recall $T^n(x)$, the n^{th} iterative of x under T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \; n = 1,2, \ldots$$

is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration starting at x_0.

Definition 6[9]: Let (X, d) be metric space. A mapping $T: X \to X$ is called weak contraction if there exists a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \text{ ----------- } (1.2)$$

Remark 7[9]: Due to symmetry of the distance, the weak contractive condition (1.2) imply implicitly includes the following dual one.

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Ty) \text{ ----------- } (1.3)$$

$\forall\, x, y \in X$

In order to check the weak contractiveness of T; it is necessary to check both (1.2) and (1.3). It is clear that any contraction mapping is also weak contraction mapping in a metric space.

Main Results: In this section, we give some fixed point theorems in b-metric spaces.

Theorem 1: Let (X, d) be a complete b-metric space with constant $s \geq 1$ and define the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ by the recursion

$$x_n = Tx_{n-1} = T^n x_0.$$

Let $T: X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4[d(y, Tx) + d(x, Ty)] \quad (1)$$

where $\lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 \leq 1$

$\forall\, x, y \in X$ then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is a unique fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined as $x_n = Tx_{n-1} = T^n x_0$,

$$n = 1, 2, 3 - - - \quad (2)$$

By (1) and (2) we obtain that

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2 d(x_{n-1}, Tx_{n-1}) + \lambda_3 d(x_n, Tx_n) + \lambda_4[d(x_n, Tx_{n-1}) \\
&\quad + d(x_{n-1}, Tx_n)] \\
&\leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2 d(x_{n-1}, x_n) + \lambda_3 d(x_n, x_{n+1}) + \lambda_4[d(x_n, x_n) \\
&\quad + d(x_{n-1}, x_{n+1})] \\
&\leq \lambda_1 d(x_{n-1}, x_n) + s\lambda_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \lambda_3 d(x_n, x_{n+1}) \\
&\quad + s\lambda_4[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&\leq \lambda_1 d(x_{n-1}, x_n) + s\lambda_2 d(x_{n-1}, x_n) + s\lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x_n, x_{n+1}) \\
&\quad + s\lambda_4 d(x_{n-1}, x_n) \\
&\leq (\lambda_1 + s\lambda_2 + s\lambda_4) d(x_{n-1}, x_n) + (s\lambda_2 + \lambda_3 + s\lambda_4) d(x_n, x_{n+1})
\end{aligned}$$

$$\Rightarrow (1 - s\lambda_2 - \lambda_3 - s\lambda_4) d(x_n, x_{n+1}) \leq (\lambda_1 + s\lambda_2 + s\lambda_4) d(x_{n-1}, x_n)$$

$$\Rightarrow d(x_n, x_{n+1}) \leq \frac{(\lambda_1 + s\lambda_2 + s\lambda_4)}{(1 - s\lambda_2 - \lambda_3 - s\lambda_4)} d(x_{n-1}, x_n) \leq k d(x_{n-1}, x_n)$$

where $k = \frac{\lambda_1 + s\lambda_2 + s\lambda_4}{1 - s\lambda_2 - \lambda_3 - s\lambda_4} \leq 1$

As $\lambda_1 + 2s\lambda_2 + \lambda_3 + 2s\lambda_4 \leq 1$

$$\lambda_1 + s\lambda_2 + s\lambda_4 \leq 1 - s\lambda_2 - \lambda_3 - s\lambda_4$$

$$\frac{\lambda_1 + s\lambda_2 + s\lambda_4}{1 - s\lambda_2 - \lambda_3 - s\lambda_4} \leq 1$$

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq k d(x_{n-1}, x_n) \\
&\leq k^2 d(x_{n-2}, x_{n-1})
\end{aligned}$$

Continuing this process, we get

$$\leq k^n d(x_0, x_1).$$

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$

$$\begin{aligned}
d(x_n, x_m) &\leq s\, d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + - - - - \\
&\leq s\, k^n\, d(x_1, x_0) + s^2 k^{n+1} d(x_1, x_0) + - - - + s^m k^{n+m-1} d(x_1, x_0) \\
&\leq s\, k^n\, d(x_1, x_0)[1 + (sk) + (sk)^2 + - - - + (sk)^{m-1}]
\end{aligned}$$

$$\leq s\, k^n\, d(x_1, x_0)\left[\frac{1-(sk)^{n-(m-1)}}{1-sk}\right].$$

When we take $m, n \to \infty$

$$lim_{n\to\infty} d(x_n, x_m) = 0.$$

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{x_n\}$ converges to $x^* \in X$.

Now we show that x^* is the unique fixed point of T.

$$d(x^*, Tx^*) \leq s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)]$$
$$\leq sd(x^*, x_{n+1}) + sd(Tx_n, Tx^*)$$
$$\leq sd(x^*, x_{n+1}) + s\lambda_1 d(x_n, x^*) + s\lambda_2 d(x_n, Tx_n) + s\lambda_3 d(x^*, Tx^*)$$
$$+ s\lambda_4[d(x^*, Tx_n) + d(x_n, Tx^*)]$$
$$\Rightarrow d(x^*, Tx^*) \leq sd(x^*, x_{n+1}) + s\lambda_1 d(x_n, x^*) + s\lambda_2 d(x_n, x_{n+1}) + s\lambda_3 d(x^*, Tx^*)$$
$$+ s\lambda_4 d(x^*, Tx_n) + s\lambda_4 d(x_n, Tx^*)$$
$$\leq sd(x^*, x_{n+1}) + s\lambda_1 d(x_n, x^*) + s^2\lambda_2 d(x_n, x^*) + s^2\lambda_2 d(x^*, x_{n+1})$$
$$+ s\lambda_3 d(x^*, Tx^*) + s\lambda_4 d(x^*, x_{n+1}) + s^2\lambda_4[d(x_n, x^*) + d(x^*, Tx^*)$$
$$(1 - s\lambda_3 - s^2\lambda_4)d(x^*, Tx^*) \leq sd(x^*, x_{n+1}) + s\lambda_1 d(x_n, x^*)$$
$$+ s^2\lambda_2 d(x^*, x_n) + s^2\lambda_2 d(x^*, x_{n+1}) + s\lambda_4 d(x^*, x_{n+1}) + s^2\lambda_4 d(x_n, x^*)$$
$$\Rightarrow (1 - s\lambda_3 - s^2\lambda_4)d(x^*, Tx^*) \leq (s + s^2\lambda_2 + s\lambda_4)d(x_{n+1}, x^*)$$
$$+ (s\lambda_1 + s^2\lambda_2 + s^2\lambda_4)d(x_n, x^*)$$
$$\Rightarrow d(x^*, Tx^*) \leq \frac{(s+s^2\lambda_2+s\lambda_4)}{(1-s\lambda_3-s^2\lambda_4)}\, d(x_{n+1}, x^*) + \frac{(s\lambda_1+s^2\lambda_2+s^2\lambda_4)}{(1-s\lambda_3-s^2\lambda_4)}\, d(x_n, x^*)$$

$d(x^*, Tx^*) \leq 0$ as $n \to \infty$. Now we show that x^* is the fixed point of T. Assume that x' is another fixed point of T, then we have $Tx' = x'$ and

$$d(x^*, x') = d(Tx^*, Tx')$$
$$\leq \lambda_1 d(x^*, x') + \lambda_2 d(x^*, Tx^*) + \lambda_3 d(x', Tx') + \lambda_4[d(x', Tx^*) + d(x^*, Tx')]$$
$$\leq \lambda_1 d(x^*, x') + \lambda_2 d(x^*, x^*) + \lambda_3 d(x', x') + \lambda_4[d(x', x^*) + d(x^*, x')]$$
$$\leq \lambda_1 d(x^*, x') + \lambda_4[d(x', x^*) + d(x^*, x')] = (\lambda_1 + 2\lambda_4)d(x^*, x')$$

which implies that $x^* = x'$.

Theorem 2 : Let (X, d) be a complete b-metric space with constant $s \geq 1$. Let $T: X \to X$ be a mapping for which there exist $\lambda_1, \lambda_2 \in [0, \frac{1}{3})$ such that

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2[d(x, Tx) + d(y, Ty)] \qquad \text{------}(3)$$
$$\forall\, x, y \in X$$

Then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is a unique fixed point of T.

Proof: Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined as $x_n = Tx_{n-1} = T^n x_0$, $n = 1, 2, 3 - - -$. By using (3)

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]$$

$$\leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$\leq (\lambda_1 + \lambda_2)d(x_{n-1}, x_n) + \lambda_2 d(x_n, x_{n+1})$$
$$\Rightarrow (1 - \lambda_2)d(x_n, x_{n+1}) \leq (\lambda_1 + \lambda_2)d(x_{n-1}, x_n)$$
$$d(x_n, x_{n+1}) \leq \left(\frac{\lambda_1 + \lambda_2}{1 - \lambda_2}\right) d(x_{n-1}, x_n)$$
$$\leq kd(x_{n-1}, x_n)$$
where $k = \frac{\lambda_1+\lambda_2}{1-\lambda_2} \leq 1$

as $\lambda_1 + 2\lambda_2 \leq 1$
$$\Rightarrow \lambda_1 + \lambda_2 \leq 1 - \lambda_2$$
$$\frac{\lambda_1 + \lambda_2}{1 - \lambda_2} \leq 1$$
$$d(x_n, x_{n+1}) \leq k^n d(x_1, x_0)$$

Thus T is a contraction mapping.

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$
$$d(x_n, x_m) \leq s\,[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)]$$
$$\leq s\, d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3})$$
$$+ \,---+ s^m d(x_{n+m-1}, x_m)$$
$$\leq s\, k^n d(x_1, x_0) + s^2 k^{n+1} d(x_1, x_0) + ---+ s^m k^{n+m-1} d(x_1, x_0)$$
$$\leq (s\, k^n)\, d(x_1, x_0)[1 + (sk) + (sk)^2 + ---+ (sk)^{m-1}]$$
$$\leq (s\, k^n) d(x_1, x_0) \left[\frac{1 - (sk)^{n-(m-1)}}{1 - sk}\right]$$

when we take $m, n \to \infty$
$$lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{x_n\}$ converges to $x^* \in X$.

Now we show that x^* is the unique fixed point of T.
$$d(x^*, Tx^*) \leq s[d(x^*, x_n) + d(x_n, Tx^*)]$$
$$d(x^*, Tx^*) \leq s[d(x^*, x_n) + d(Tx_{n-1}, Tx^*)$$
$$\leq sd(x^*, x_n) + sd(Tx_{n-1}, Tx^*)$$
$$\leq sd(x^*, x_n) + s[\lambda_1 d(x_{n-1}, x^*)] + s\lambda_2[d(x_{n-1}, Tx_{n-1}) + d(x^*, Tx^*)]$$
$$\leq sd(x^*, x_n) + s\lambda_1 d(x_{n-1}, x^*) + s\lambda_2 d(x_{n-1}, x_n) + s\lambda_2 d(x^*, Tx^*)$$
$$\Rightarrow d(x^*, Tx^*) \leq sd(x_n, x^*) + s\lambda_1 d(x_{n-1}, x^*) + s^2\lambda_2[d(x_{n-1}, x^*) + d(x^*, x_n)]$$
$$+ s\lambda_2 d(x^*, Tx^*)$$
$$\Rightarrow d(x^*, Tx^*) \leq sd(x_n, x^*) + s\lambda_1 d(x_{n-1}, x^*) + s^2\lambda_2 d(x_{n-1}, x^*)$$
$$+ s^2\lambda_2 d(x_n, x^*) + s\lambda_2 d(x^*, Tx^*)$$
$$\Rightarrow (1 - s\lambda_2)d(x^*, Tx^*) \leq sd(x_n, x^*) + s\lambda_1 d(x_{n-1}, x^*) + s^2\lambda_2 d(x_{n-1}, x^*)$$
$$+ s^2\lambda_2 d(x_n, x^*)$$

$$lim_{n\to\infty} d(x^*, Tx^*) = 0$$
$$\text{i.e } Tx^* = x^*$$

Now we show that x^* is the unique fixed point of T. Assume that x' is another fixed point of T, then we have T$x' = x'$ and

$$d(x^*, x') = d(Tx^*, Tx')$$
$$\leq \lambda_1 d(x^*, x') + \lambda_2[d(x^*, Tx^*) + d(x', Tx')]$$
$$d(x^*, x') \leq \lambda_1 d(x^*, x')$$
$$(1 - \lambda_1)\, d(x^*, x') \leq 0$$

which implies that $x^* = x'$

This completes the proof.

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