Some Fixed Point Results in Dislocated Quasi-Metric Space

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Abstract:

In this paper, we prove some fixed point theorems for different contraction mappings in dislocated quasi-metric space. The results presented in this paper extend and improve several well-known results in the literature.

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Key Words: Dislocated quasi-metric, fixed point.

1. Introduction

Banach (1922) proved fixed point theorem for contraction mappings in complete metric space. It is well-known as a Banach fixed point theorem. It has many applications in various branches of mathematics such as differential equation, integral equation, fractals, mathematical economics etc. Kannan [7] proved a fixed point theorem for new type of contraction mappings called Kannan mappings in a complete metric space. Lj. B. Ciric [5] gave a generalization of Banach contraction principle in metric space. In 2008, Aage & Salunke [3] proved some fixed point theorems for Kannan mappings and generalized contraction mappings in dislocated quasi-metric space.

In this paper, we prove some fixed point theorems for different contraction mappings in dislocated quasi-metric space.

2. Preliminary

Definition 2.1[4]. Let X be a nonempty set and let $d: X \times X \to [0, \infty]$ be a function satisfying following conditions:

(i) $d(x, y) = d(y, x) = 0$ implies $x = y$

(ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a dislocated quasi-metric on X. If d satisfies $d(x, x) = 0$, then it is called a quasi-metric on X. If d satisfies $d(x, y) = d(y, x)$ then it is called dislocated metric.

Definition 2.2[4]. A sequence $\{x_n\}$ in dq-metric space (dislocated quasi-metric space) (X, d) is called Cauchy sequence if for given $\varepsilon > 0\ \exists n_0 \in N$ such that $\forall\, m, n \geq n_0$, implies $d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$

i.e. $\min\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon$.

Definition 2.3[4]. A sequence $\{x_n\}$ dislocated quasi-converges to x if

$$lim_{n\to\infty} d(x_n, x) = lim_{n\to\infty} d(x, x_n) = 0$$

In this case x is called a dq-limit of $\{x_n\}$ and we write $x_n \to x$.

Definition 2.4[4]. A dq-metric space (X, d) is called complete if every Cauchy sequence in it is a dq-convergent.

Definition 2.5[4]. Let (X, d_1) and (Y, d_2) be dq-metric spaces and let $f: X \to Y$ be a function. Then f is continuous to $x_0 \in X$, if for each sequence $\{x_n\}$ which is $d_1 - q$ convergent to x_0, the sequence $\{f(x_n)\}$ is $d_2 - q$ convergent to $f(x_0)$ in Y.

Definition 2.6[4]. Let (X, d) be a dq-metric space. A map $T: X \to X$ is called contraction if there exists $0 \leq \lambda < 1$ such that

$d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$.

Lemma 2.7. Let (X, d) be a dq-metric space and let $f: X \to X$ is a contraction function then $\{(f^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.

3. Main Results

Theorem 3.1. Let (X, d) be a complete dq-metric space and let $T: X \to X$ be a continuous mapping satisfying the following condition

$$d(T\boldsymbol{x}, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] \quad \text{--------(3.1)}$$

where a, b are non negative, which may depends on both x and y, such that $Sup\{a + 2b: x, y \in X\} < 1$. Then T has unique fixed point.

Proof : Let $\{x_n\}$ be a sequence in X, defined as follows

Let $x_0 \in X, T(x_0) = x_1, T(x_1) = x_2, - - - - - T(x_n) = x_{n+1}$

Now consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq ad(x_{n-1}, x_n) + b[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]$$

$$= ad(x_{n-1}, x_n) + b[d(x_{n-1,} x_n) + d(x_n, x_{n+1})]$$

$$= ad(x_{n-1}, x_n) + bd(x_{n-1,} x_n) + b\, d(x_n, x_{n+1})$$

$$\leq (a + b)d(x_{n-1}, x_n) + bd(x_n, x_{n+1})$$

$$d(x_n, x_{n+1}) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \lambda\, d(x_{n-1}, x_n)$$

$$where\ \lambda \le \frac{(a+b)}{(1-b)}$$

Similarly we have $d(x_{n-1}, x_n) \le \lambda\, d(x_{n-2}, x_{n-1})$

In this way, we get

$$d(x_n, x_{n+1}) \le \lambda^n\, d(x_0, x_1).$$

Since $0 \le \lambda < 1$ so far $n \to \infty$, we have $d(x_n, x_{n+1}) \to \infty$. Similarly we show that $d(x_{n+1}, x_n) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence in the complete dislocated quasi-metric space X. So there is a paint $t_0 \in X$ such that $x_n \to t_0$. Since T is continuous we have

$$T(t_0) = lim_{n\to\infty}\, T(x_n) = lim_{n\to\infty}\, x_{n+1} = t_0.$$

Thus $T(\,t_0) = t_0$. Thus T has a fixed point.

Uniqueness : Let x be a fixed point of T. Then by given condition we have

$$d(x,x) = d(Tx,Tx)$$

$$\le ad(x,x) + b[d(x,Tx) + d(x,Tx)]$$

$$\le ad(x,x) + 2bd(x,x)$$

$$d(x,x) \le (a + 2b)d(x,x)$$

which is true only if $d(x,x) = 0$, since $0 \le (a + 2b) < 1$ and $d(x,x) = 0$.

Thus $d(x,x) = 0$, if x is fixed point of T.

Now let x, y be fixed point of T. That is $Tx = x, Ty = y$, then by given condition,

we have

$$d(x,y) = d(Tx,Ty)$$

$$\le ad(x,y) + b[d(x,Tx) + d(y,Ty)]$$

$$\le ad(x,y) + b[d(x,x) + d(y,y)]$$

$$d(x,y) \le ad(x,y)$$

Similarly we have

$$d(y,x) \le ad(y,x)$$

Hence $|d(x,y) - d(y,x)| \le a|d(x,y) - d(y,x)|$

which implies that $d(x,y) = d(y,x), 0 \le a < 1$.

Again from (3.1)

$$d(x,y) \le ad(x,y)$$

which gives $d(x,y) = 0$ since $0 \le a < 1$.

Further $d(x, y) = d(y, x) = 0$ gives $x = y$.

Hence fixed point is unique.

Theorem 3.2. Let (X, d) be a complete dq-metric space and let $T: X \to X$ be a continuous mapping satisfying the following condition

$$d(T\boldsymbol{x}, Ty) \leq ad(x, y) + b[d(x, Ty) + d(y, Tx)] \qquad \text{--------(3.2)}$$

where a, b are non negative, which may depends on both x and y, such that $a + 2b < 1$. Then T has unique fixed point.

Proof : Let $\{x_n\}$ be a sequence in X, defined as follows:

Let $x_0 \in X, T(x_0) = x_1, T(x_1) = x_2, - - - - - T(x_n) = x_{n+1}$

Now consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq ad(x_{n-1}, x_n) + b[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]$$

$$= ad(x_{n-1}, x_n) + b[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]$$

$$\leq ad(x_{n-1}, x_n) + b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$\leq (a + b)d(x_{n-1}, x_n) + bd(x_n, x_{n+1})$$

$$d(x_n, x_{n+1}) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n)$$

$$\leq \lambda\, d(x_{n-1}, x_n)$$

$$where\ \lambda \leq \frac{(a+b)}{(1-b)}.$$

Rest of the proof of this theorem is same as theorem 3.1.

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