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KANNAN ITERATED FUNCTION SYSTEM

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Abstract

It is well known that the mapping condition given by Kannan[1] is more lenient than contraction condition. The purpose of this note is to introduce Kannan Iterated Function System which will cover a larger range of mappings. We also prove the Collage theorem for the Kannan Iterated Function System.

Keywords: K-Iterated Function System; Iterated Function System; Kannan Mapping; Compact Set; Contraction Mapping.

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1. INTRODUCTION

One of the most eye-catching applications of contraction mapping lies in fractal theory. Fractal geometry has been found to be a very effective mean for modeling the infinite details found in nature. The re-exploration of fractal geometry is usually traced back to the publication of the book "*The Fractal Geometry of Nature*"[2] by the IBM mathematician Benoit B. Mandelbrot. Iterated Function System is a method of constructing fractals, which consists of a set of maps that explicitly list the similarities of the shape. Though the formal name Iterated Function Systems or IFS was coined by Barnsley and Demko[3] in 1985, the basic concept is usually attributed to Hutchinson.[4] However Vrscay[5] have traced the idea back to Williams,[6] who studied fixed points of infinite composition of contractive maps.

The idea of fractals and especially IFS has been extensively studied because of its variety of application in image compression, simulations and so on. In 1994, Gröller[7] showed that use of nonlinear function increases the flexibility when defining an IFS. Study in this field was further carried on by Frame and Angers.[8] The concept of multifunction has been applied by Torre and others.[9–11]

In this note we introduce a new Iterated Function System namely "Kannan Iterated Function System" or "KIFS", which will cover a larger range of mappings.

2. ITERATED FUNCTION SYSTEM

Let X denotes a complete metric space with distance function d and T be a mapping from X into itself. Then T is called a *contraction mapping* if there is a constant $0 \leq s < 1$ such that

$$d(T(x), T(y)) \leq sd(x, y). \tag{1}$$

The constant s is called the contractivity factor for T.

Polish mathematician S. Banach proved a very important result, regarding contraction mapping in 1922, known as Banach Contraction principle.[12]

Theorem 2.1. *Let $T : X \to X$ be a contraction mapping, with contractivity factor 's', on a complete metric space (X, d). Then T possesses exactly one fixed point $x^* \in X$. Moreover, for any point $x \in X$, the sequence $\{T_n(x) : n = 0, 1, 2, \ldots\}$ converges to x^*. That is $\lim_{n\to\infty} T^n(x) = x^*$, for each $x \in X$.*

IFS generally employ contractive maps over a complete metric space (X, d), where the Banach's celebrated result mentioned above guarantees the existence and uniqueness of the fixed point known as "attractor". The main property of contraction mapping which is used in IFS is given by the following lemma:

Lemma 2.2. *Let $T : X \to X$ be a contraction mapping, with contractivity factor $'s'$, on a complete metric space (X, d). Then T is continuous.*

We now discuss certain definitions required to understand iterated function system. Let (X, d) be a complete metric space and $\mathcal{H}(X)$ denote the space whose points are the compact subsets of X known as Hausdroff space, other than the empty set. Let $x, y \in X$ and let $A, B \in \mathcal{H}(X)$. Then

(1) *distance from the point x to the set B* is defined as
$$d(x, B) = \min\{d(x, y) : y \in B\},$$
(2) *distance from the set A to the set B* is defined as
$$d(A, B) = \max\{d(x, B) : x \in A\},$$
(3) *Hausdroff distance from the set A to the set B* is defined as
$$h(A, B) = d(A, B) \vee d(B, A).$$

Then the function $h(d)$ is the metric defined on the space $\mathcal{H}(X)$.

Note: Throughout this paper the notation $u \vee v$ means the maximum and $u \wedge v$ denotes the minimum of the pair of real numbers u and v.

In IFS, the contractive maps act on the members of Hausdroff space, i.e., the compact subsets of X. Thus, an iterated function system is defined as follows:

A (*hyperbolic*) *iterated function system* consists of a complete metric space (X, d) together with a finite set of continuous contraction mappings $T_n : X \to X$ with respect to contractivity factor s_n, for $n = 1, 2, 3, \ldots, N$.

Thus, the following theorem was given by Barnsley[13]:

Theorem 2.3. *Let $\{X : T_n, n = 1, 2, 3, \ldots, N\}$ be a iterated function system with contractivity factor s. Then the transformation $\mathcal{W} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $\mathcal{W}(B) = \bigcup_{n=1}^{N} T_n(B)$ for all $B \in \mathcal{H}(X)$,* *is a contraction mapping on the complete metric space $(\mathcal{H}(X), h(d))$ with contractivity factor s. That is*

$$h(\mathcal{W}(B), \mathcal{W}(C)) \leq sh(B, C).$$

Its unique fixed point, which is also called an attractor, $A \in \mathcal{H}(X)$, obeys

$$A = \mathcal{W}(A) = \bigcup_{n=1}^{N} T_n(A),$$

and is given by $A = \lim_{n\to\infty} \mathcal{W}^{on}(B)$ for any $B \in \mathcal{H}(X)$. W^{on} denotes the n-fold composition of W.

The contraction mappings used in IFS are typically affine maps. The iteration dynamics associated with affine maps is not very interesting but when the action of a system of contraction mappings is considered the result is quite remarkable.

For example in two dimensions, let $X = [0, 1]^2$, consider the IFS formed by the following three affine maps

$$f_1(x, y) = \left(\frac{1}{2}x, \frac{1}{2}y\right),$$

$$f_2(x, y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right),$$

$$f_3(x, y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right).$$

All the above mappings have contractivity factor $\frac{1}{2}$ and their fixed point lies on the vertices of an equilateral triangle $(0, 0), (1, 0)$ and $(1, \frac{\sqrt{3}}{2})$. The resulting attractor is known as "*Sierpinski gasket*".

3. K-ITERATED FUNCTION SYSTEM

In this section, we shall try to explore the possibility of improvement in IFS by replacing contraction condition by a more general condition known as Kannan condition. In 1969, Kannan[1] introduced a mapping, which was an improvement over contraction mapping, known as *Kannan mapping* defined as follows:

If there exists a number α, $0 < \alpha < 1/2$, such that, for all $x, y \in X$,

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))]. \quad (2)$$

Then T is called a *Kannan mapping*. Let us name α as *K-contractivity factor* of Kannan mapping T.

On the basis of definition of (hyperbolic) iterated function system given by Barnsley,[13] we now introduce K-iterated function system:

A K-iterated function system consists of a complete metric space (X, d) together with a finite set of Kannan mappings $T_n : X \to X$ with K-contractivity factor α_n, for $n = 1, 2, 3, \ldots, N$.

First of all we state and prove the two propositions which will establish a relation between T^m : $m = 1, 2, \ldots, N$ and α; and uniqueness of fixed point of T if it exists, respectively.

Proposition 3.1. *Let $T : X \to X$ be a Kannan mapping, with K-contractivity factor 'α', on a metric space (X, d) and $x \in X$. Then T satisfies the following condition*:

$$d(T^m(x), T^{m+1}(x)) \leq \left(\frac{\alpha}{1-\alpha}\right)^m d(x, T(x)).$$

Moreover, $\lim_{m\to\infty} d(T^m(x), T^{m+1}(x)) = 0$

Proof. Since T is a Kannan contraction mapping, we have

$$d(T^m(x), T^{m+1}) \leq \alpha(d(T^{m-1}(x), T^m(x)) + d(T^m(x), T^{m+1}(x))).$$

It follows that

$$\begin{aligned} d(T^m(x), T^{m+1}(x)) &\leq \frac{\alpha}{1-\alpha} d(T^{m-1}(x), T^m(x)) \\ &\leq \frac{\alpha}{1-\alpha}\left[\frac{\alpha}{1-\alpha} d(T^{m-2}(x), T^{m-1}(x))\right] \\ &\cdots \\ &\cdots \\ &\leq \left(\frac{\alpha}{1-\alpha}\right)^m d(x, T(x)). \end{aligned}$$

Taking limit as $m \to \infty$, we have

$$\lim_{m\to\infty} d(T^m(x), T^{m+1}(x)) \leq \lim_{m\to\infty} \left(\frac{\alpha}{1-\alpha}\right)^m d(x, T(x)).$$

Therefore, $\lim_{m\to\infty} d(T^m(x), T^{m+1}(x)) = 0$, since $\frac{\alpha}{1-\alpha} < 1$. □

Proposition 3.2. *Let $T : X \to X$ be a Kannan mapping, with K-contractivity factor 'α', on a metric space (X, d). If T has a fixed point, then it is unique.*

Proof. On the contrary, let x^* and y^* be two fixed points of T. Then $x^* = T(x^*)$, $y^* = T(y^*)$, and

$$\begin{aligned} d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\leq \alpha[d(x^*, T(x^*)) + d(y^*, T(y^*))] \\ &= \alpha[d(x^*, x^*) + d(y^*, y^*)] \\ &= 0, \end{aligned}$$

Therefore, $x^* = y^*$. □

Next, we prove the following proposition, which shows the principle underlying the Collage theorem for Kannan mapping.

Proposition 3.3. *Let $T : X \to X$ be a continuous Kannan mapping on a complete metric space (X, d) with contractivity factor and let $x* \in X$ be the fixed point of T. Then*

$$d(x, x^*) \leq \left(\frac{1-\alpha}{1-2\alpha}\right) d(x, T(x)), \quad \forall\, x \in X.$$

Proof. For $x \in X$, we have $\lim_{n\to\infty} T^n(x) = x^*$. Taking the point $a \in X$ as fixed, we know that the distance function $d(a, b)$ is continuous at the point $b \in X$, we conclude

$$\begin{aligned} d(x, x^*) &= d\Big(x, \lim_{n\to\infty} T^n(x)\Big) \\ &= \lim_{n\to\infty} d(x, T^n(x)) \\ &\leq \lim_{n\to\infty} \sum_{m=1}^{n} d(T^{m-1}(x), T^m(x)) \\ &\leq \lim_{n\to\infty} d(x, T(x)) \\ &\quad \times \left(1 + \frac{\alpha}{1-\alpha} + \cdots + \left(\frac{\alpha}{1-\alpha}\right)^{n-1}\right) \\ &\leq \left(1 - \frac{\alpha}{1-\alpha}\right)^{-1} d(x, T(x)). \end{aligned}$$

This completes the proof. □

Using Propositions 3.1 and 3.2 we now prove the following theorem which is an extension of Contraction mapping theorem for Kannan mapping.

Theorem 3.4. *Let $T : X \to X$ be a Kannan mapping, with K-contractivity factor $'\alpha'$, on a complete metric space (X, d). Then T possesses exactly one fixed point $x^* \in X$ and moreover for any point $x \in X$, the sequence $\{T^n(x) : n = 0, 1, 2, \ldots\}$ converges to x^*. That is $\lim_{n\to\infty} T^n(x) = x^*$, for each $x \in X$.*

Proof. Let $x \in X$. Since T is a Kannan mapping with K-contractivity factor α, we have

$$d(T^m(x), T^{m+1}(x)) \leq \left(\frac{\alpha}{1-\alpha}\right)^m d(x, T(x)), \quad \forall\, m = 0, 1, 2, \ldots.$$

Then, for any fixed $x \in X$, we get

$$d(T^n(x), T^m(x)) \leq s^{m\wedge n} d(x, T^{|n-m|}(x)), \tag{3}$$

where $m, n = 0, 1, 2, \ldots$ and $s := \frac{\alpha}{1-\alpha}$. In particular, let us take $k = |n - m|$, for $k = 0, 1, 2, \ldots,$ we have

$$\begin{aligned} d(x, T^k(x)) &\leq d(x, T(x)) + d(T(x), T^2(x)) \\ &\quad + \cdots + d(T^{k-1}(x), T^k(x)) \\ &\leq (1 + s + s^2 + \cdots + s^{k-1}) d(x, T(x)) \\ &\leq \left(\frac{1-s^k}{1-s}\right) d(x, T(x)). \end{aligned}$$

On substituting in Eq. (3), we obtain

$$d(T^n(x), T^m(x)) \leq \frac{s^{m\wedge n}(1-s^k)}{1-s} d(x, T(x)),$$

it immediately follows that $\{T^n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence. Since X is a complete metric space, this Cauchy sequence has a limit $x^* \in X$, and we have

$$\lim_{n\to\infty} T^n(x) = x^*. \tag{4}$$

Now to prove that x^* is a fixed point of T we see that

$$\begin{aligned} d(x^*, T(x^*)) &\leq d(x^*, T^n(x)) + d(T^n(x), T(x^*)) \\ &\leq d(x^*, T^n(x)) + \alpha[d(T^{n-1}(x), T^n(x)) \\ &\quad + d(x^*, T(x^*))]. \end{aligned}$$

Taking limit as $n \to \infty$, on considering Eqn. (4) and Proposition (3.1), we get $d(x^*, T(x^*)) \leq (1 + \alpha) d(x^*, T(x^*))$. Hence $x^* = T(x^*)$. By Proposition 3.2, x^* is unique. This completes the proof. □

Lemma 3.5. *Let $T : X \to X$ be a continuous Kannan mapping on the metric space (X, d) with K-contractivity factor 'α'. Then $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $T(B) = \{T(x) : x \in B\}$) for every $B \in \mathcal{H}(X)$ is a Kannan mapping on $(\mathcal{H}(X), h(d))$ with contractivity factor α.*

Proof. Since T is a continuous mapping, therefore by Lemma 2 of Ref. 13, T maps $\mathcal{H}(X)$ into itself.

Let $B, C \in \mathcal{H}(X)$. Then

$$\begin{aligned} h(T(B),T(C)) &= d(T(B),T(C)) \vee d(T(C),T(B)) \\ &\leq \alpha\{[d(B,T(B)) + d(C,T(C))] \\ &\quad \vee [d(C,T(C)) + d(B,T(B))]\} \\ &= \alpha[d(B,T(B)) + d(C,T(C))] \\ &\leq \alpha[h(B,T(B)) + h(C,T(C))]. \end{aligned}$$

Therefore,

$$h(T(B),T(C)) \leq \alpha[h(B,T(B)) + h(C,T(C))].$$

This completes the proof. □

Lemma 3.6. *Let (X,d) be a metric space. Let $T_n : n = 1,2,3,\ldots,N$ be continuous Kannan mappings on $(\mathcal{H}(X),h)$. Let the K-contractivity factor for T_n be denoted by α_n for each n. Define $\mathcal{T} : \mathcal{H}(X) \to \mathcal{H}(X)$ by $\mathcal{T}(B) = T_1(B) \cup T_2(B) \cup \cdots \cup T_N(B) = \bigcup_{n=1}^{N} T_n(B)$ for each $B \in \mathcal{H}(X)$. Then $\mathcal{T}$ is a Kannan mapping with K-contractivity factor $\alpha = max\{\alpha_n : n = 1,2,\ldots,N\}$.*

Proof. We shall prove the theorem using mathematical induction method using the properties of metric h. For $N = 1$, the statement is obviously true. Now for $N = 2$, we see that

$$\begin{aligned} &h(\mathcal{T}(B),\mathcal{T}(C)) \\ &\quad = h(T_1(B) \cup T_2(B), T_1(C) \cup T_2(C)) \\ &\quad \leq h(T_1(B),T_1(C)) \vee h(T_2(B),T_2(C)) \\ &\quad \leq \alpha_1[h(B,T_1(B)) + h(C,T_1(C))] \\ &\qquad \vee \alpha_2[h(B,T_2(B)) + h(C,T_2(C))] \\ &\quad \leq (\alpha_1 \vee \alpha_2)[\{h(B,T_1(B)) \vee h(B,T_2(B))\} \\ &\qquad + \{h(C,T_1(C)) \vee h(C,T_2(C))\}] \\ &\quad = \alpha[h(B,T_1(B) \cup T_2(B)) \\ &\qquad + h(C,T_1(C) \cup T_2(C))]. \end{aligned}$$

Therefore,

$$h(\mathcal{T}(B),\mathcal{T}(C)) \leq \alpha[h(B,\mathcal{T}(B)) + h(C,\mathcal{T}(C))].$$

By the condition of mathematical induction Lemma 3.6 is proved. □

Thus, from all the above results and the definition of K-iterated function system (KIFS), we are in the position to present the following theorem for KIFS.

Theorem 3.7. *Let $\{X;(T_0),T_1,T_2,\ldots,T_N\}$, where T_0 is the condensation mapping be a K-iterated function system with K-contractivity factor α. Then the transformation $\mathcal{T} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $\mathcal{T}(B) = \bigcup_{n=1}^{N} T_n(B)$ for all $B \in \mathcal{H}(X)$ is a continuous Kannan mapping on the complete metric space $(\mathcal{H}(X),h(d))$ with contractivity factor α. Its unique fixed point, which is also called an attractor, $A \in \mathcal{H}(X)$, obeys*

$$A = \mathcal{T}(A) = \bigcup_{n=1}^{N} A,$$

and is given by $A = \lim_{n\to\infty} \mathcal{T}^{on}(B)$ for any $B \in \mathcal{H}(X)$.

Based on above mathematical formulation of Proposition 3.3, we can prove the following *Collage theorem* for KIFS.

Theorem 3.8. *Let (X,d) be a complete metric space. Let $L \in \mathcal{H}(X)$ be given and $\epsilon \geq 0$ be given. Choose an KIFS $\{X;(T_0),T_1,T_2,\ldots,T_N\}$, where T_0 is the condensation mapping with K-contractivity factor α, so that*

$$h\left(L, \bigcup_{n=0,n=1}^{N} T_n(L)\right) \leq \epsilon.$$

Then

$$h(L,A) \leq \epsilon\frac{1-\alpha}{1-2\alpha},$$

where A is the attractor of the KIFS. Equivalently,

$$h(L,A) \leq \left(\frac{1-\alpha}{1-2\alpha}\right) \cdot h\left(L, \bigcup_{n=0,n=1}^{N} T_n(L)\right),$$

for all $L \in \mathcal{H}(X)$.

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