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COUPLED FIXED POINT RESULTS WITH C-DISTANCE IN CONE METRIC SPACES

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Abstract. In this paper, we prove some coupled fixed point results under c-distance satisfying certain contractive condition in cone metric spaces. Our results generalize the result of Fadail and Ahmad (2012). Example is given to show the usability of our results.
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1. Introduction

Cone metric spaces were introduced by Huang and Zhang [9]. Then, several fixed and common fixed point theorems in cone meteric spaces were proved in [1-7,23,25] and the references contained therein.

In 2006, Bhaskar and Lakshmikantham [15] considered the concept of coupled fixed point theorems in partially ordered metric spaces. Afterward, many authors generalized and proved several common coupled fixed and coupled fixed point theorems in ordered metric and ordered cone metric spaces (see [16],[17], [20], [22],[24], [25]).

Recently, Cho et al. [19] introduced a new concept of a c-distance; a cone version of a w-distance of Kada et al. [13], in cone metric spaces and proved some fixed point theorems in partially ordered cone metric spaces under c-distance. Then sintunavarat et al. [18] proved fixed point theorems and extended the Banach contraction theorem on c-distance. Subsequently, several authors have studied and proved fixed point theorems under generalized distance in cone metric spaces, e.g ([12],[21],[24],[25]). In 2012, Cho et al. [20] proved coupled fixed point theorems under weak contractions by using the concept of c-distance.

In this manuscript, we prove some coupled fixed point results with c-distance in cone metric space. Our results generalize the result of Fadail and Ahmad [24].

2. Preliminaries

Definition 2.1[9]: Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if

a): P is closed, non-empty, and $P \neq \{\theta\}$;
b): $a, b \in R, a, b \geqslant 0$,$x, y \in P \Rightarrow ax + by \in P$;
c): If $x \in P$ and $-x \in P$ then $x = \theta$.

θ denote to the zero element in E.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by

$x \preceq y \Leftrightarrow y - x \in P.$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in int\, P$ (where $int\, P$ is the interior of P). The cone P is called normal if there is a number $K > 0$ such that, for all $x, y \in E$, we have

$\theta \preceq x \preceq y \Rightarrow \| x \| \leqslant K \| y \| .$

The least positive number satisfying the above is called the normal constant of P.

Definition 2.2[9]: Let X be a non-empty set and the mapping $d : X \times X \to E$ satisfies;

(i) $\theta \preceq d\,(x, y)$**:** for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$**:** for all $x, y \in X$;

(iii) $d(x,z) \preceq d(x,y) + d(y,z)$**:** for all $x,y,z \in X$.

Then, d is called a cone metric on X, and (X,d) is called a cone metric space.

Definition 2.3[9]: Let (X,d) be a cone metric space and $\{x_n\}$ be a sequence in X and $x \in X$.

(1): for all $c \in E$ with $\theta \ll c$, if there exists a positive integer $\mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > \mathbb{N}$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$,

(2): for all $c \in E$ with $\theta \ll c$, if there exists a positive integer $\mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in X, and

(3): a cone metric space (X,d) is called complete if every Cauchy sequence in X is convergent.

Lemma 2.4[7]:

(1) If E be a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

(2) If $c \in int\, P, \theta \preceq a_n\, and\, a_n \to \theta$, then there exists a positive integer $\mathbb{N}$ such that $a_n \ll c$ for all $n \geq \mathbb{N}$.

Next, we give the notion of c-distance on a cone metric space (X,d) of Wang and Guo in [14] which is a generalization of w-distance of Kada et. al [13] with some properties.

Definition 2.5[14]: Let (X,d) be cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(1) $\theta \preceq q(x,y)$ for all $x,y \in X$,

(2) $q(x,y) \preceq q(x,y) + q(y,z)$ for all $x,y,z \in X$,

(3) for each $x \in X$ and $n \geq 1$ if $q(x,y_n) \preceq u$ for some $u = u_x \in P$, then $q(x,y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$, and

(4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Example 2.6[14]: Let $E = \mathbb{R}\, and\, P = \{x \in E : x \geq 0\}$. Let $X = [0,\infty)$ and define a mapping $d : X \times X \to E$ by $d(x,y) = (|x-y|)$ for all $x,y \in X$. Then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = y$ for for all $x,y \in X$. Then q is a c-distance on X.

Lemma 2.7[14]: Let (X,d) be a cone metric space and q is a c-distance on X. Let$\{x_n\}$and $\{y_n\}$ be sequences in X and $x,y,z \in X$. Suppose that u_n is a sequences in P converging to θ. Then the following hold:

(1) If $q(x_n,y) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $y = z$.

(2) If $q(x_n,y_n) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $\{y_n\}$ converges to z.

(3) If $q(x_n,x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $q(y,x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

Remark 2.8[14]:

(1) $q(x,y) = q(y,x)$ does not necessarily for all $x,y \in X$.

(2) $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

3. Main Results

In this section, we prove some coupled fixed point theorems under c-distance in the context of cone metric spaces.

Theorem 3.1. Let (X,d) be a complete cone metric space and q is a c-distance on X. Let $F : X \times X \to X$ be a mapping and suppose that there exists mappings $k,l,r,n : X \times X \to [0,1)$ such that the following hold:

(a) $k(F(x,y),F(u,v)) \leq k(x,y), l(F(x,y),F(u,v)) \leq l(x,y)$,
$r(F(x,y),F(u,v)) \leq r(x,y)\, and\, n(F(x,y),F(u,v)) \leq n(x,y)$ for all $x,y,u,v \in X$;

(b) $k(x,y)=k(y,x), l(x,y)=l(y,x), r(x,y)=r(y,x)$ and $n(x,y)=n(y,x)$ for all $x,y\in X$;

(c) $(k+l+r+2n)(x,y)<1$ for all $x,y\in X$;

(d) $q(F(x,y),F(u,v))\preceq k(x,y)q(x,u)+l(x,y)q(x,F(x,y))$
$+r(x,y)q(u,F(u,v))$
$+n(x,y)[q(F(x,y),u)+q(F(u,v),x)]$

for all $x,y,u,v\in X$.

Then F has a coupled fixed point $(x^\star,y^\star)\in X\times X$. Further, if $x_1=F(x_1,y_1)$, and $y_1=F(y_1,x_1)$ then $q(x_1,x_1)=\theta\, and\, q(y_1,y_1)=\theta$. Moreover, the coupled fixed point is unique and is of the form $(x^\star,x^\star)$ for some $x^\star\in X$.

Proof. Take $x_0,y_0\in X$, Set $x_1=F(x_0,y_0), y_1=F(y_0,x_0)$,
$x_2=F(x_1,y_1), y_2=F(y_1,x_1),------ x_{n+1}=F(x_n,y_n),\ y_{n+1}=F(y_n,x_n)$. Then we have

$q(x_n,x_{n+1})=q(F(x_{n-1},y_{n-1}),F(x_n,y_n))$
$\preceq k(x_{n-1},y_{n-1})q(x_{n-1},x_n)+l(x_{n-1},y_{n-1})q(x_{n-1},F(x_{n-1},y_{n-1}))+r(x_{n-1},y_{n-1})q(x_n,F(x_n,y_n))$
$+n(x_{n-1},y_{n-1})[q(F(x_{n-1},y_{n-1}),x_n)+q(F(x_n,y_n),x_{n-1}))]$
$=k(F(x_{n-2},y_{n-2}),F(y_{n-2},x_{n-2}))q(x_{n-1},x_n)+l(F(x_{n-2},y_{n-2}),F(y_{n-2},x_{n-2}))q(x_{n-1},x_n)$
$+r(F(x_{n-2},y_{n-2}),F(y_{n-2},x_{n-2}))q(x_n,x_{n+1})$

$+n(F(x_{n-2},y_{n-2}),F(y_{n-2},x_{n-2}))[q(x_n,x_n)+q(x_{n+1},x_{n-1})]$

$\preceq k(x_{n-2},y_{n-2})q(x_{n-1},x_n)+l(x_{n-2},y_{n-2})q(x_{n-1},x_n)$

$+r(x_{n-2},y_{n-2})q(x_n,x_{n+1})+n(x_{n-2},y_{n-2})[q(x_{n+1},x_n)+q(x_n,x_{n-1})]$

$\preceq ...$

$\preceq k(x_0,y_0)q(x_{n-1},x_n)+l(x_0,y_0)q(x_{n-1},x_n)+$

$r(x_0,y_0)q(x_n,x_{n+1})+n(x_0,y_0)[q(x_{n+1},x_n)+q(x_{n-1},x_n)]-----(3.1)$

and similarly

$q(y_n, y_{n+1}) = q(F(y_{n-1}, x_{n-1}), F(y_n, x_n))$

$\preceq k(y_{n-1}, x_{n-1})q(y_{n-1}, y_n) + l(y_{n-1}, x_{n-1})q(y_{n-1}, F(y_{n-1}, x_{n-1})) + r(y_{n-1}, x_{n-1})q(y_n, F(y_n, x_n))$

$+n(y_{n-1}, x_{n-1})[q(F(y_{n-1}, x_{n-1}), y_n) + q(F(y_n, x_n), y_{n-1})]$

$= k(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(y_{n-1}, y_n) + l(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(y_{n-1}, y_n)$

$+r(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(y_n, y_{n+1})$

$+n(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))[q(y_n, y_n) + q(y_{n+1}, y_{n-1})]$

$\preceq k(y_{n-2}, x_{n-2})q(y_{n-1}, y_n) + l(y_{n-2}, x_{n-2})q(y_{n-1}, y_n)$

$+r(y_{n-2}, x_{n-2})q(y_n, y_{n+1}) + n(y_{n-2}, x_{n-2})[q(y_{n+1}, y_n) + q(y_{n-1}, y_n)]$

$\preceq \ldots$

$\preceq k(x_0, y_0)q(y_{n-1}, y_n) + l(x_0, y_0)q(y_{n-1}, y_n) + r(x_0, y_0)q(y_n, y_{n+1})$

$+n(x_0, y_0)[q(y_{n+1}, y_n) + q(y_{n-1}, y_n)] - - - - - (3.2)$

Put $q_n = q(x_n, x_{n+1}) + q(y_n, y_{n+1})$. Then from (3.1) and (3.2), we have

$q_n = q(x_n, x_{n+1}) + q(y_n, y_{n+1})$

$\preceq (k(x_0, y_0) + l(x_0, y_0) + n(x_0, y_0))(q(x_{n-1}, x_n) + q(y_{n-1}, y_n))$

$+(r(x_0, y_0) + n(x_0, y_0))(q(x_n, x_{n+1}) + q(y_n, y_{n+1}))$

$= hq_{n-1}$

$\ldots$

$\preceq h^n q_0 - - - - - (3.3)$

where $h = \frac{k(x_0,y_0)+l(x_0,y_0)+n(x_0,y_0)}{1-r(x_0,y_0)-n(x_0,y_0)} < 1.$

Let $m > n \geq 1$. It follows that

$q(x_n, x_m) \preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + - - - - + q(x_{m-1}, x_m)$

$q(y_n, y_m) \preceq q(y_n, y_{n+1}) + q(y_{n+1}, y_{n+2}) + - - - - + q(y_{m-1}, y_m)$

By adding above two, we have

$q(x_n, x_m) + q(y_n, y_m) \preceq q_n + q_{n+1} + - - - - - - - q_{m-1}$

$\preceq h^n q_0 + h^{n+1} q_0 + - - - - - - + h^{m-1} q_0$

$= (h^n + h^{n+1} + - - - - - - + h^{m-1}) q_0$

$\preceq \frac{h^n}{1-h} q_0 - - - - - (3.4)$

From (3.4) we have

$q(x_n, x_m) \preceq \frac{h^n}{1-h} q_0 - - - - - (3.5)$

and also $q(y_n, y_m) \preceq \frac{h^n}{1-h} q_0 - - - - - (3.6)$

Thus, Lemma 2.7(3) shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X. Since X is complete, there exists $x^\star$ and $y^\star \in X$ such that $x_n \to x^\star$ and $y_n \to y^\star$ as $n \to \infty$. By Definition 2.5(3) we have the following:

$q(x_n, x^\star) \preceq \frac{h^n}{1-h} q_0 - - - - - (3.7)$

and also $q(y_n, y^\star) \preceq \frac{h^n}{1-h} q_0 - - - - - (3.8)$

On the other hand,

$q(x_n, F(x^\star, y^\star)) = q(F(x_{n-1}, y_{n-1}).F(x^\star, y^\star))$

$\preceq k(x_{n-1}, y_{n-1})q(x_{n-1}, x^\star) + l(x_{n-1}, y_{n-1})q(x_{n-1}, F(x_{n-1}, y_{n-1}))$

$+r(x_{n-1}, y_{n-1})q(x^\star, F(x^\star, y^\star))$

$+n(x_{n-1}, y_{n-1})[q(F(x_{n-1}, y_{n-1}), x^\star) + q(F(x^\star, y^\star), x_{n-1})]$

$= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(x_{n-1}, x^\star)$

$+l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(x_{n-1}, x_n)$
$+r(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(x^\star, x_{n+1})$

$+n(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))[q(x_n, x^\star) + q(x_{n+1}, x_{n-1})]$

$\preceq k(x_{n-2}, y_{n-2})q(x_{n-1}, x^\star) + l(x_{n-2}, y_{n-2})q(x_{n-1}, x^\star)$

$+r(x_{n-2}, y_{n-2})q(x^\star, x_{n+1})$

$+n(x_{n-2}, y_{n-2})[q(x_{n-1}, x^\star) + q(x_{n-1}, x^\star)]$

$\preceq ...$

$\preceq k(x_0, y_0)q(x_{n-1}, x^\star) + l(x_0, y_0)q(x_{n-1}, x^\star)$

$+r(x_0, y_0)q(x^\star, x_{n+1}) + n(x_0, y_0)[q(x_{n+1}, x^\star) + q(x_{n-1}, x^\star)]$

$= [k(x_0, y_0) + l(x_0, y_0) + n(x_0, y_0)]q(x_{n-1}, x^\star)$

$+[r(x_0, y_0) + n(x_0, y_0)]q(x_{n+1}, x^\star)$

$\preceq \frac{k(x_0,y_0)+l(x_0,y_0)+n(x_0,y_0)}{1-r(x_0,y_0)-n(x_0,y_0)} q(x_{n-1}, x^\star)$

$= h\frac{h^{n-1}}{1-h} q_0$

$= \frac{h^n}{1-h} q_0 - - - - - (3.9)$

By Lemma 2.7(1), (3.7) and (3.9), we have $x^\star = F(x^\star, y^\star)$. By similar way we have $y^\star = F(y^\star, x^\star)$. Therefore $(x^\star, y^\star)$ is a coupled fixed point of F. Suppose that $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then we have

$q(x_1, x_1) = q(F(x_1, y_1), F(x_1, y_1))$

$\preceq k(x_1, y_1)q(x_1, x_1) + l(x_1, y_1)q(x_1, F(x_1, y_1))$

$+r(x_1, y_1)q(x_1, F(x_1, y_1))$

$+n(x_1, y_1)[q(F(x_1, y_1), x_1) + q(F(x_1, y_1), x_1)]$

$= k(x_1, y_1)q(x_1, x_1) + l(x_1, y_1)q(x_1, x_1)$

$+r(x_1, y_1)q(x_1, x_1) + n(x_1, y_1)[q(x_1, x_1) + q(x_1, x_1)] - - - - - (3.10)$

and also

$q(y_1, y_1) \preceq k(x_1, y_1)q(y_1, y_1) + l(x_1, y_1)q(y_1, y_1) + r(x_1, y_1)q(y_1, y_1)$

$+n(x_1, y_1)[q(y_1, y_1) + q(y_1, y_1)] - - - - - (3.11)$

which implies that

$q(x_1, x_1) + q(y_1, y_1) \preceq (k(x_1, y_1) + l(x_1, y_1) + r(x_1, y_1) + 2n(x_1, y_1))$

$(q(x_1, x_1) + q(y_1, y_1)) - - - - - (3.12)$

Since $(k + l + r + 2n)(x_1, y_1) < 1$, Lemma 2.4(1) shows that $q(x_1, x_1) + q(y_1, y_1) = \theta$. But $q(x_1, x_1) \succeq \theta$ and $q(y_1, y_2) \succeq \theta,\ hence\, q(x_1, x_1) = \theta\ and\, q(y_1, y_1) = \theta.$

Now, we show that the uniqueness of coupled fixed point. Suppose that there is another coupled fixed point (x', y') then we have

$$q(x^\star, x^{'}) = q(F(x^\star, y^\star), F(x^{'}, y^{'}))$$

$$\preceq k(x^\star, y^\star)q(x^\star, x^{'}) + l(x^\star, y^\star)q(x^\star, F(x^\star, y^\star)) + r(x^\star, y^\star)q(x^{'}, F(x^{'}, y^{'}))$$

$$+n(x^\star, y^\star)[q(F(x^\star, y^\star), x^{'}) + q(F(x^{'}, y^{'}), x^\star)]$$

$$= k(x^\star, y^\star)q(x^\star, x^{'}) + n(x^\star, y^\star)[q(x^\star, x^{'}) + q(x^{'}, x^\star)] - - - - - (3.13)$$

Similarly

$$q(y^\star, y^{'}) \preceq k(x^\star, y^\star)q(y^\star, y^{'}) + n(x^\star, y^\star)[q(y^\star, y^{'}) + q(y^{'}, y^\star)] - - - - - (3.14)$$

which implies that

$$q(x^\star, x^{'}) + q(y^\star, y^{'}) \preceq (k(x^\star, y^\star) + 2n(x^\star, y^\star))(q(x^\star, x^{'}) + q(y^\star, y^{'}))$$

$$= (k + 2n)(x^\star, y^\star)(q(x^\star, x^{'}) + q(y^\star, y^{'})) - - - - - (3.15)$$

Since $(k + 2n)(x^\star, y^\star) < 1$, Lemma 2.4(1) shows that $q(x^\star, x^{'}) + q(y^\star, y^{'}) = \theta$. But $q(x^\star, x^{'}) \succeq \theta$ and $q(y^\star, y^{'}) \succeq \theta$. Hence $q(x^\star, x^{'}) = \theta$ and $q(y^\star, y^{'}) = \theta$. Also we have $q(x^\star, x^\star) = \theta$ and $q(y^\star, y^\star) = \theta$. Hence Lemma 2.7(1) shows that $x^\star = x^{'}$and $y^\star = y^{'}$, which implies that $(x^\star, y^\star) = (x^{'}, y^{'})$. Similarly, we prove that $x^\star = y^{'}$ and $y^\star = x^{'}$. Hence $x^\star = y^\star$. Therefore, the coupled fixed point is unique and is of the form $(x^\star, x^\star)$ for some $x^\star \in X$.

From above Theorem we have the following Corollaries.

Corollary 3.2. Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:

$$q(F(x, y), F(u, v)) \preceq kq(x, u) + lq(x, F(x, y)) + rq(u, F(u, v))$$
$$+n[q(F(x, y), u) + q(F(u, v), x)] - - - - - (3.16)$$

for all $x, y, u, v \in X$, where k, l, r, n are nonnegative real constants with $k+l+r+2n < 1$. Then F has a coupled fixed point $(x^\star, y^\star) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and

$y_1 = F(y_1, x_1)$ then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form $(x^\star, x^\star)$ for some $x^\star \in X$.

By putting $l = 0\, and\, r = 0$ in Corollary 3.2, we get the following Corollaries.

Corollary 3.3. Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:

$q(F(x, y), F(u, v)) \preceq kq(x, u) + n[q(F(x, y), u) + q(F(u, v), x)] - - - - - (3.17)$

for all $x, y, u, v \in X$, where k, n are nonnegative real constants with $k + 2n < 1$. Then F has a coupled fixed point $(x^\star, y^\star) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$ then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form $(x^\star, x^\star)$ for some $x^\star \in X$.

Corollary 3.4. Let (X, d) be complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:

$q(F(x, y), F(u, v)) \preceq kq(x, u) + n[q(F(x, y), x) + q(F(u, v), u)] - - - - - (3.18)$

for all $x, y, u, v \in X$, where k, n are nonnegative real constants with $k + 2n < 1$. Then F has a coupled fixed point $(x^\star, y^\star) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$ then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form $(x^\star, x^\star)$ for some $x^\star \in X$.

Theorem 3.5. Let (X, d) be a complete cone metric space, and q is a c-distance on X. Let $F : X \times X \to X$ be a mapping and suppose that there exists mappings $k, l, r : X \times X \to [0, 1)$ such that the following hold:

(a) $k(F(x, y), F(u, v)) \leq k(x, y), l(F(x, y), F(u, v)) \leq l(x, y)$ and $r(F(x, y), F(u, v)) \leq r(x, y)$ for all $x, y, u, v \in X$;

(b) $k(x, y) = k(y, x), l(x, y) = l(y, x)\, and\, r(x, y) = r(y, x)$ for all $x, y \in X$;

(c) $(k + 2l + 2r)(x, y) < 1$, for all $x, y \in X$, and

(d) $q(F(x,y),F(u,v)) \preceq k(x,y)q(x,u) + l(x,y)[q(x,F(u,v)) + q(u,F(x,y))]$

$+r(x,y)[q(x,F(x,y)) + q(u,F(u,v))]$ for all $x,y,u,v \in X$;

Then F has a coupled fixed point $(x^\star, y^\star) \in X \times X$. Further, if $x_1 = F(x_1,y_1)$ and $y_1 = F(y_1,x_1)$ then $q(x_1,x_1) = \theta$ and $q(y_1,y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form $(x^\star, x^\star)$ for some $x^\star \in X$.

Proof. Choose $x_0, y_0 \in X,\ Set\, x_1 = F(x_0,y_0), y_1 = F(y_0,x_0),$
$x_2 = F(x_1,y_1), y_2 = F(y_1,x_1), - - - - - x_{n+1} = F(x_n,y_n), y_{n+1} = F(y_n,x_n).$
Then we have the following:

$q(x_n,x_{n+1}) = q(F(x_{n-1},y_{n-1}),F(x_n,y_n))$

$\preceq k(x_{n-1},y_{n-1})q(x_{n-1},x_n)$

$+l(x_{n-1},y_{n-1})[q(x_{n-1},F(x_n,y_n)) + q(x_n,F(x_{n-1},y_{n-1}))]$

$+r(x_{n-1},y_{n-1})[q(x_{n-1},F(x_{n-1},y_{n-1})) + q(x_n,F(x_n,y_n))]$

$= k(F(x_{n-2},y_{n-2}),F(y_{n-2},x_{n-2}))q(x_{n-1},x_n)$

$+l(F(x_{n-2},y_{n-2}),F(y_{n-2},x_{n-2}))[q(x_{n-1},x_{n+1}) + q(x_n,x_n)]$

$+r(F(x_{n-2},y_{n-2}),F(y_{n-2},x_{n-2}))[q(x_{n-1},x_n) + q(x_n,x_{n+1})]$

$\preceq k(x_{n-2},y_{n-2})q(x_{n-1},x_n) + l(x_{n-2},y_{n-2})[q(x_{n-1},x_n) + q(x_n,x_{n+1})]$

$+r(x_{n-2},y_{n-2})[q(x_{n-1},x_n) + q(x_n,x_{n+1})]$

$\preceq ...$

$\preceq k(x_0,y_0)q(x_{n-1},x_n) + l(x_0,y_0)[q(x_{n-1},x_n) + q(x_n,x_{n+1})]$

$+r(x_0,y_0)[q(x_{n-1},x_n) + q(x_n,x_{n+1})]$

Then we have

$q(x_n, x_{n+1}) \preceq \frac{k(x_0,y_0)+l(x_0,y_0)+r(x_0,y_0)}{1-l(x_0,y_0)-r(x_0,y_0)} q(x_{n-1}, x_n)$

$= hq(x_{n-1}, x_n)$

$\preceq h^2 q(x_{n-2}, x_{n-1})$

$\preceq ...$

$\preceq h^n q(x_0, x_1)$

$i.e\, q(x_n, x_{n+1}) \preceq h^n q(x_0, x_1) ----(3.19)$

where $h = \frac{k(x_0,y_0)+l(x_0,y_0)+r(x_0,y_0)}{1-l(x_0,y_0)-r(x_0,y_0)} < 1.$

Similarly we have

$q(y_n, y_{n+1}) \preceq h^n q(y_0, y_1) ----- (3.20)$

Put $q_n = q(x_n, x_{n+1}) + q(y_n, y_{n+1})$. Then we have

$q_n = q(x_n, x_{n+1}) + q(y_n, y_{n+1})$

$\preceq h^n(q(x_0, x_1) + q(y_0, y_1)) = h^n q_0$

i.e $q_n \preceq h^n q_0 ----- (3.21)$

where $h = \frac{k(x_0,y_0)+l(x_0,y_0)+r(x_0,y_0)}{1-l(x_0,y_0)-r(x_0,y_0)} < 1.$

Let $m > n \geq 1$. Then it follows that

$q(x_n, x_m) \preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + ------ + q(x_{m-1}, x_m)$

and $q(y_n, y_m) \preceq q(y_n, y_{n+1}) + q(y_{n+1}, y_{n+2}) + - - - - - - + q(y_{m-1}, y_m)$,

Then we have

$q(x_n, x_m) + q(y_n, y_m) \preceq q_n + q_{n+1} + - - - - - - + q_{m-1}$

$\preceq h^n q_0 + h^{n+1} q_0 + - - - - - - + h^{m-1} q_0$

$= (h^n + h^{n+1} + - - - - - - + h^{m-1}) q_0$

$\preceq \frac{h^n}{1-h} q_0 - - - - - (3.22)$

From (3.22), we have

$q(x_n, x_m) \preceq \frac{h^n}{1-h} q_0 - - - - - (3.23)$

and also $q(y_n, y_m) \preceq \frac{h^n}{1-h} q_0 - - - - - (3.24)$

Thus, Lemma 2.7(3) shows that $\{x_n\}$and $\{y_n\}$ are Cauchy sequence in X. Since X is complete, there exists $x^\star$ and $y^\star \in X$ such that $x_n \to x^\star$ and $y_n \to y^\star$ as $n \to \infty$. By Definition 2.5(3) we have the following:

$q(x_n, x^\star) \preceq \frac{h^n}{1-h} q_0 - - - - - (3.25)$

and also $q(y_n, y^\star) \preceq \frac{h^n}{1-h} q_0 - - - - - (3.26)$

On the other hand,

$q(x_n, F(x^\star, y^\star)) = q(F(x_{n-1}, y_{n-1}), F(x^\star, y^\star))$

$\preceq k(x_{n-1}, y_{n-1}) q(x_{n-1}, x^\star)$

$+l(x_{n-1}, y_{n-1})[q(x_{n-1}, F(x^\star, y^\star)) + q(x^\star, F(x_{n-1}, y_{n-1}))]$

$+r(x_{n-1}, y_{n-1})[q(x_{n-1}, F(x_{n-1}, y_{n-1})) + q(x^\star, F(x^\star, y^\star))]$

$= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(x_{n-1}, x^\star)$

$+l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))[q(x_{n-1}, x_{n+1}) + q(x^\star, x_n)]$

$+r(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))[q(x_{n-1}, x_n) + q(x^\star, x_{n+1})]$

$\preceq k(x_{n-2}, y_{n-2})q(x_{n-1}, x^\star) + l(x_{n-2}, y_{n-2})[q(x_{n-1}, x^\star) + q(x^\star, x_{n+1})]$

$+r(x_{n-2}, y_{n-2})[q(x_{n-1}, x_n) + q(x^\star, x_{n+1})]$

$\preceq ...$

$\preceq k(x_0, y_0)q(x_{n-1}, x^\star) + l(x_0, y_0)[q(x_{n-1}, x^\star) + q(x^\star, x_{n+1})]$

$+r(x_0, y_0)[q(x_{n-1}, x^\star) + q(x^\star, x_{n+1})]$

$= (k(x_0, y_0) + l(x_0, y_0) + r(x_0, y_0))q(x_{n-1}, x^\star)$

$+(l(x_0, y_0) + r(x_0, y_0))q(x^\star, x_{n+1})$

$\preceq \frac{k(x_0,y_0)+l(x_0,y_0)+r(x_0,y_0)}{1-l(x_0,y_0)-r(x_0,y_0)} q(x_{n-1}, x^\star)$

$= h\frac{h^{n-1}}{1-h}q_0 = \frac{h^n}{1-h}q_0 - - - - - (3.27)$

By Lemma 2.7(1), (3.25) and (3.27), we have $x^\star = F(x^\star, y^\star)$. By similar way we have $y^\star = F(y^\star, x^\star)$. Therefore $(x^\star, y^\star)$ is a coupled fixed point of F.

Suppose that $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then we have

$q(x_1, x_1) = q(F(x_1, y_1), F(x_1, y_1))$

$\preceq k(x_1,y_1)q(x_1,x_1)+l(x_1,y_1)[q(x_1,F(x_1,y_1))+q(x_1,F(x_1,y_1))]$

$+r(x_1,y_1)[q(x_1,F(x_1,y_1))+q(x_1,F(x_1,y_1))]$

$= k(x_1,y_1)q(x_1,x_1)+l(x_1,y_1)[q(x_1,x_1)+q(x_1,x_1)]$

$+r(x_1,y_1)[q(x_1,x_1)+q(x_1,x_1)] -----(3.28)$

and also

$q(y_1,y_1)=q(F(y_1,x_1),F(y_1,x_1))$

$\preceq k(y_1,x_1)q(y_1,y_1)+l(y_1,x_1)[q(y_1,y_1)+q(y_1,y_1)]$

$+r(y_1,x_1)[q(y_1,y_1)+q(y_1,y_1)]$

$= k(x_1,y_1)q(y_1,y_1)+l(x_1,y_1)[q(y_1,y_1)+q(y_1,y_1)]$

$+r(x_1,y_1)[q(y_1,y_1)+q(y_1,y_1)] -----(3.29)$

which implies that

$q(x_1,x_1)+q(y_1,y_1)\preceq (k(x_1,y_1)+2l(x_1,y_1)+2r(x_1,y_1))$

$(q(x_1,x_1)+q(y_1,y_1))$

$=(k+2l+2r)(x_1,y_1)(q(x_1,x_1)+q(y_1,y_1)) -----(3.30)$

Since $(k+2l+2r)(x_1,y_1)<1$, Lemma 2.4(1) shows that $q(x_1,x_1)+q(y_1,y_1)=\theta$. But $q(x_1,x_1)\succeq\theta$ and $q(y_1,y_2)\succeq\theta,\ hence\, q(x_1,x_1)=\theta\, and\, q(y_1,y_1)=\theta.$

Finally, suppose that there is another coupled fixed point $(x^{'},y^{'})$ then we have

$q(x^{\star},x')=q(F(x^{\star},y^{\star}),F(x^{'},y^{'}))$

$$\preceq k(x^\star, y^\star)q(x^\star, x^{'}) + l(x^\star, y^\star)[q(x^\star, F(x^{'}, y^{'})) + q(x^{'}, F(x^\star, y^\star))]$$

$$+r(x^\star, y^\star)[q(x^\star, F(x^\star, y^\star)) + q(x^{'}, F(x^{'}, y^{'}))]$$

$$= k(x^\star, y^\star)q(x^\star, x^{'}) + l(x^\star, y^\star)[q(x^\star, x^{'}) + q(x^{'}, x^\star)]$$

$$+r(x^\star, y^\star)[q(x^\star, x^\star) + q(x^{'}, x^{'})]$$

$$\preceq k(x^\star, y^\star)q(x^\star, x^{'}) + 2l(x^\star, y^\star)q(x^\star, x^{'}) - - - - - (3.31)$$

and also

$$q(y^\star, y^{'}) = q(F(y^\star, x^\star), F(y^{'}, x^{'}))$$

$$\preceq k(y^\star, x^\star)q(y^\star, y^{'}) + l(y^\star, x^\star)[q(y^\star, F(y^{'}, x^{'})) + q(y^{'}, F(y^\star, x^\star))]$$

$$+r(y^\star, x^\star)[q(y^\star, F(y^\star, x^\star)) + q(y^{'}, F(y^{'}, x^{'}))]$$

$$= k(y^\star, x^\star)q(y^\star, y^{'}) + l(y^\star, x^\star)[q(y^\star, y^{'}) + q(y^{'}, y^\star)]$$

$$+r(y^\star, x^\star)[q(y^\star, y^\star) + q(y^{'}, y^{'})]$$

$$\preceq k(x^\star, y^\star)q(y^\star, y^{'}) + 2l(x^\star, y^\star)q(y^\star, y^{'}) - - - - - (3.32)$$

which implies that

$$q(x^\star, x^{'}) + q(y^\star, y^{'}) \preceq (k(x^\star, y^\star) + 2l(x^\star, y^\star))(q(x^\star, x^{'}) + q(y^\star, y^{'}))$$

$$= (k + 2l)(x^\star, y^\star)(q(x^\star, x^{'}) + q(y^\star, y^{'})) - - - - - (3.33)$$

Since $(k + 2l)(x^\star, y^\star) < 1$, Lemma 2.4(1) shows that $q(x^\star, x^{'}) + q(y^\star, y^{'}) = \theta$. But $q(x^\star, x^{'}) \succeq \theta$ and $q(y^\star, y^{'}) \succeq \theta$. Hence $q(x^\star, x^{'}) = \theta$ and $q(y^\star, y^{'}) = \theta$. Also we have $q(x^\star, x^\star) = \theta$ and $q(y^\star, y^\star) = \theta$. Hence Lemma 2.7(1) shows that $x^\star = x^{'}$and $y^\star = y^{'}$, which implies that $(x^\star, y^\star) = (x^{'}, y^{'})$. Similarly, we prove that $x^\star = y^{'}$ and $y^\star = x^{'}$.

Hence $x^\star = y^\star$. Therefore, the coupled fixed point is unique and is of the form $(x^\star, x^\star)$ for some $x^\star \in X$.

Corollary 3.6. Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:

$$q(F(x,y),F(u,v)) \preceq kq(x,u) + l[q(x,F(u,v)) + q(u,F(x,y))]$$
$$+r[q(x,F(x,y)) + q(u,F(u,v))] - - - - - (3.34)$$

for all $x, y, u, v \in X$, where k, l, r are nonnegative constants with $k+2l+2r < 1$. Then F has a coupled fixed point $(x^\star, y^\star) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$ then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form $(x^\star, x^\star)$ for some $x^\star \in X$.

Example 3.7. Let $E = \mathbb{R}^2$ *and* $P = \{(x, y) \in E : x, y \geq 0\}$. Let $X = [0, 1)$ and define a mapping $d : X \times X \to E$ by

$d(x, y) = (|x - y|, |x - y|)$ for all $x, y \in X$. Then (X, d) is complete cone metric space, see[6]. Define a mapping $q : X \times X \to E$ by $q(x, y) = (y, y)$ for all $x, y \in X$. Then q is a c-distance on X. In fact, first three conditions of Definition 2.5 are immediate. Let $c \in E$ with $\theta \ll c$ and put $e = c/2$. If $q(z, x) \ll e$ and $q(z, y) \ll e$, then we have

$$d(x,y) = (|x-y|, |x-y|)$$

$$\preceq (x+y, x+y)$$

$$= (x,x) + (y,y)$$

$$= q(z,x) + q(z,y)$$

$$\ll e + e$$

$$= c. - - - - - (3.35)$$

This shows that fourth condition of Definition 2.5 holds. Therefore, q is c-distance on X. Define the mapping $F : X \times X \to X$ by $F(x, y) = \frac{x+y}{8}$ for all $(x, y) \in X \times X$. Then we have

$$q(F(x, y), F(u, v)) = (F(u, v), F(u, v))$$

$$= (\tfrac{u+v}{8}, \tfrac{u+v}{8}) = (\tfrac{u}{8}, \tfrac{u}{8}) + (\tfrac{v}{8}, \tfrac{v}{8})$$

$$= (\tfrac{1}{8})(u, u) + (\tfrac{1}{8})(v, v)$$

$$\preceq (\tfrac{1}{4})(u, u) + (\tfrac{2}{4})(v, v)$$

$$= (\tfrac{1}{4})(u, u) + (\tfrac{1}{4})[(v, v) + (v, v)]$$

$$= (\tfrac{1}{4})(u, u) + (\tfrac{1}{4})[q(x, v) + q(u, v)]$$

$$= (\tfrac{1}{4})(u, u) + (\tfrac{1}{4})[q(v, x) + q(v, u)]$$

$$= (\tfrac{1}{4})(u, u) + (\tfrac{1}{4})[(x, x) + (u, u)]$$

$$= (\tfrac{1}{4})(u, u) + (\tfrac{1}{4})[q(F(x, y), x) + q(F(u, v), u)]$$

$$\preceq kq(x, u) + n[q(F(x, y), x) + q(F(u, v), u)]$$

with $k = n = \frac{1}{4}$, and $k+2n = \frac{3}{4} < 1$. Therefore, the condition of Corollary 3.4 are satisfied, and then F has a unique coupled fixed $(x, y) = (0, 0)\, and\, F(0, 0) = 0\, with\, q(0, 0) = 0$.

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